

Dirac equation in curved spacetimes using coordinate-free notation

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June 8, 2010

Abstract

The Dirac equation in curved spacetimes is formulated using coordinate-free notation. A Lagrangean density which corresponds to the subject equation is presented. It is shown that the subject equation is invariant under a local rotation of the coframe. It is shown that the current is independent of the local orientation of the coframe and that it is conserved. It is shown that the subject equation has an equivalent formulation which uses the Christoffel gamma. A calculation of the stress-energy tensor in teleparallel gravity replicates the result that a version of the teleparallel equivalent of general relativity is inconsistent in the presence of a Dirac field.

1 Introduction

The Dirac equation in curved spacetimes has previously been studied [1] [2] [3]. A coordinate-free form different from the one in this article is in [4]. In this article, the Dirac equation in curved spacetimes is formulated using coordinate-free notation. The layout of this article is: In section 2, I show that the Dirac equation in curved spacetimes is derived from the Dirac Lagrangean. In section 3, I show that the subject equation is invariant under a local rotation of the coframe. In section 4, I show that the current is conserved. In section 5, I show that the subject equation has an equivalent formulation which uses the Christoffel gamma. Appendix 1 contains a proof

that ω has the same values as the rotational part of the torsion-free Cartan connection $(\theta, \omega) : T(\mathcal{M}^4) \rightarrow \mathfrak{euc}(3, 1)$. In Appendix 2, a calculation of the stress-energy tensor of the Dirac field in teleparallel gravity replicates the result that a version of the teleparallel equivalent of general relativity is inconsistent in the presence of a Dirac field.

For a curved spacetimes Dirac equation, we usually use a coframe 1-form $\theta : T(\mathcal{M}^4) \rightarrow \mathbb{R}^4$ and a dual vector field $\theta^j(v_k) = \delta^j_k$. I use,

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (1)$$

$$\gamma^0 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \gamma^j = \begin{bmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{bmatrix} \quad (2)$$

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} \quad (4)$$

$$\bar{\Psi} = \Psi^\dagger \gamma^0 \quad (5)$$

$$S^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] \quad (6)$$

$$\epsilon^{0123} = \epsilon_{0123} \quad (7)$$

Latin indices are Lorentzian and Greek indices are spacetime indices. Latin indices are raised or lowered using the Minkowski metric:

$$A^a = \eta^{ab} A_b \quad (8)$$

The coframe θ can be expressed using the coordinate 1-forms,

$$\theta^a = \theta^a_\mu dx^\mu \quad (9)$$

and the metric is,

$$g_{\mu\nu} = \eta_{ab} \theta^a_\mu \theta^b_\nu \quad (10)$$

Equation (10) is 10 equations for 16 components, so for a given metric, θ has 6 degrees of freedom.

The Dirac equation in curved spacetimes is,

$$\left(\gamma^a v_a + ie\gamma^a A(v_a) - \frac{1}{4}\gamma^a \gamma^b \gamma^c \omega_{bc}(v_a) + m \right) \Psi = 0 \quad (11)$$

$$\omega_{bc}(v_a) = \frac{1}{2} \left(d\theta_a(v_b, v_c) + d\theta_b(v_a, v_c) - d\theta_c(v_a, v_b) \right) \quad (12)$$

In differential geometry style [5], vectors are represented as derivatives, so $v_a \Psi$ is an equivalent notation to $d\Psi(v_a)$. ω is antisymmetric in its two indices:

$$\omega_{bc} = -\omega_{cb} \quad (13)$$

so these two terms are equal:

$$-\frac{1}{4}\gamma^a \gamma^b \gamma^c \omega_{bc}(v_a) = -\frac{1}{2}\gamma^a S^{bc} \omega_{bc}(v_a) \quad (14)$$

e is the electric charge of the particle. A is the electromagnetic potential represented as a 1-form. An alternative formulation for the Dirac equation in curved spacetimes is,

$$\left(\gamma^a v_a^\mu \partial_\mu + ie\gamma^a A(v_a) - \frac{1}{2}\gamma^a S^{bc} v_a^\mu v_b^\nu \left(\theta_{c\rho} \Gamma_{\nu\mu}^\rho - \partial_\mu \theta_{c\nu} \right) + m \right) \Psi = 0 \quad (15)$$

As a side note, ω has the same values as the rotational part of the torsion-free Cartan connection $(\theta, \omega) : T(\mathcal{M}^4) \rightarrow \mathfrak{euc}(3, 1)$ because ω is antisymmetric and follows the equation,

$$d\theta^a - \omega^a_b \wedge \theta^b = 0 \quad (16)$$

The proof is in Appendix 1.

In this article, the Hodge dual is defined as,

$$* (\theta_{i_0} \wedge \dots \wedge \theta_{i_{k-1}}) = \frac{1}{(n-k)!} \epsilon_{i_0 \dots i_{k-1} j_0 \dots j_{n-k-1}} \theta^{j_0} \wedge \dots \wedge \theta^{j_{n-k-1}} \quad (17)$$

This means that,

$$*1 = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \quad (18)$$

$$**1 = * (\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3) \quad (19)$$

$$= * (\eta^{00} \eta^{11} \eta^{22} \eta^{33} \theta_0 \wedge \theta_1 \wedge \theta_2 \wedge \theta_3) \quad (20)$$

$$= - * (\theta_0 \wedge \theta_1 \wedge \theta_2 \wedge \theta_3) \quad (21)$$

$$= -1 \quad (22)$$

The current is usually defined as

$$j = -e\bar{\Psi}\gamma^a\Psi*\theta_a \quad (23)$$

In curved spacetimes, we would like the current to be conserved,

$$dj = 0 \quad (24)$$

We would also like that if Ψ_1 is a solution of the curved spacetimes Dirac equation with coframe θ_1 , and θ_2 another coframe for the same metric, then for a solution Ψ_2 of the curved spacetimes Dirac equation with coframe θ_2 to exist that has the same current:

$$j_1 = -e\bar{\Psi}_1\gamma^a\Psi_1*\theta_{1a} \quad (25)$$

$$j_2 = -e\bar{\Psi}_2\gamma^a\Psi_2*\theta_{2a} \quad (26)$$

$$j_1 = j_2 \quad (27)$$

2 Lagrangean density

The Lagrangean density which corresponds to the Dirac equation in curved spacetimes is,

$$\begin{aligned} \mathfrak{L} = & -\frac{1}{2}dA \wedge *dA + \\ & + i\bar{\Psi} \left(\gamma^a d\Psi \wedge *\theta_a + ie\bar{\Psi}\gamma^a A \wedge *\theta_a \Psi - \frac{1}{4}\gamma^a\gamma^b\gamma^c\omega_{bc} \wedge *\theta_a \Psi + m\Psi *1 \right) \end{aligned} \quad (28)$$

To show that this Lagrangean density corresponds to the Dirac equation in curved spacetimes, I proceed in the usual manner, setting up a region of spacetime and a small variation of the field $\delta\Psi$ which vanishes on the region boundary. The variation of the Lagrangean density is,

$$\begin{aligned} \delta\mathfrak{L} = & i\delta\bar{\Psi} \left(\gamma^a d\Psi \wedge *\theta_a + ie\gamma^a A \wedge *\theta_a \Psi - \frac{1}{4}\gamma^a\gamma^b\gamma^c\omega_{bc} \wedge *\theta_a \Psi + m\Psi *1 \right) + \\ & + i\bar{\Psi} \left(\gamma^a d\delta\Psi \wedge *\theta_a + ie\gamma^a A \wedge *\theta_a \delta\Psi - \frac{1}{4}\gamma^a\gamma^b\gamma^c\omega_{bc} \wedge *\theta_a \delta\Psi + m*\delta\Psi \right) \end{aligned} \quad (29)$$

I have,

$$d(\bar{\Psi}\gamma^a\delta\Psi*\theta_a) = d\bar{\Psi}\wedge*\theta_a\gamma^a\delta\Psi + \bar{\Psi}\gamma^a d\delta\Psi\wedge*\theta_a + \bar{\Psi}\gamma^a\delta\Psi d*\theta_a \quad (30)$$

so,

$$\bar{\Psi}\gamma^a d\delta\Psi\wedge*\theta_a = d(\bar{\Psi}\gamma^a\delta\Psi*\theta_a) - d\bar{\Psi}\wedge*\theta_a\gamma^a\delta\Psi - \bar{\Psi}\gamma^a\delta\Psi d*\theta_a \quad (31)$$

Substituting this, I get,

$$\begin{aligned} \delta\mathfrak{L} = i\delta\bar{\Psi}\left(\gamma^a d\Psi\wedge*\theta_a + ie\gamma^a A\wedge*\theta_a\Psi - \frac{1}{4}\gamma^a\gamma^b\gamma^c\omega_{bc}\wedge*\theta_a\Psi + m\Psi*1\right) + \\ + i\left(-d\bar{\Psi}\wedge*\theta_a\gamma^a - \bar{\Psi}\gamma^a d*\theta_a + ie\bar{\Psi}\gamma^a A\wedge*\theta_a - \frac{1}{4}\bar{\Psi}\gamma^a\gamma^b\gamma^c\omega_{bc}\wedge*\theta_a + \right. \\ \left. + m\bar{\Psi}*1\right)\delta\Psi + d(i\bar{\Psi}\gamma^a\delta\Psi*\theta_a) \end{aligned} \quad (32)$$

Using equations (151) - (154), it can be gathered that,

$$\gamma^a\gamma^b\gamma^c\omega_{bc}(v_a)*1 = -\gamma^c\gamma^b\gamma^a\omega_{bc}(v_a)*1 - 4\gamma^b d\theta^a(v_b, v_a)*1 \quad (33)$$

$$d*\theta_0 = d(\theta^1\wedge\theta^2\wedge\theta^3) \quad (34)$$

$$= d\theta^1\wedge\theta^2\wedge\theta^3 - \theta^1\wedge d\theta^2\wedge\theta^3 + \dots \quad (35)$$

$$= d\theta^1(v_0, v_1)*1 + d\theta^2(v_0, v_2)*1 + \dots \quad (36)$$

$$d*\theta_a = d\theta^b(v_a, v_b)*1 \quad (37)$$

$$\gamma^a\gamma^b\gamma^c\omega_{bc}\wedge*\theta_a = -\gamma^c\gamma^b\gamma^a\omega_{bc}\wedge*\theta_a - 4\gamma^a d*\theta_a \quad (38)$$

Substituting this, I get,

$$\begin{aligned} \delta\mathfrak{L} = i\delta\bar{\Psi}\left(\gamma^a d\Psi\wedge*\theta_a + ie\gamma^a A\wedge*\theta_a\Psi - \frac{1}{4}\gamma^a\gamma^b\gamma^c\omega_{bc}\wedge*\theta_a\Psi + m\Psi*1\right) + \\ + i\left(-d\bar{\Psi}\wedge*\theta_a\gamma^a + ie\bar{\Psi}\gamma^a A\wedge*\theta_a + \frac{1}{4}\bar{\Psi}\gamma^c\gamma^b\gamma^a\omega_{bc}\wedge*\theta_a + m\bar{\Psi}*1\right)\delta\Psi + \\ + d(i\bar{\Psi}\gamma^a\delta\Psi*\theta_a) \end{aligned} \quad (39)$$

Comparing with the Dirac equation in curved spacetimes and its Hermitean conjugate (equation (147)),

$$\begin{aligned} \delta\mathfrak{L} = 2\text{Re}\left(i\delta\bar{\Psi}\left(\gamma^a d\Psi\wedge*\theta_a + ie\gamma^a A\wedge*\theta_a\Psi - \frac{1}{4}\gamma^a\gamma^b\gamma^c\omega_{bc}\wedge*\theta_a\Psi + \right. \right. \\ \left. \left. + m\Psi*1\right)\right) + d(i\bar{\Psi}\gamma^a\delta\Psi*\theta_a) \end{aligned} \quad (40)$$

it can be seen that the field equation from this Lagrangean density is the same equation. Because the variation in the Lagrangean density is real up to a 4-divergence, the following Lagrangean density is equivalent up to a 4-divergence:

$$\begin{aligned} \mathfrak{L} = & -\frac{1}{2}dA \wedge *dA + \\ & + \text{Re} \left(i\bar{\Psi} \left(\gamma^a d\Psi \wedge *\theta_a + ie\bar{\Psi}\gamma^a A \wedge *\theta_a \Psi - \frac{1}{4}\gamma^a \gamma^b \gamma^c \omega_{bc} \wedge *\theta_a \Psi + m\Psi *1 \right) \right) \end{aligned} \quad (41)$$

Using,

$$\text{Re}(z) = \frac{z + z^\dagger}{2} \quad (42)$$

$$D_\mu = \partial_\mu + ieA_\mu - \frac{1}{2}S^{bc}\omega_{bc}(\partial_\mu) \quad (43)$$

$$\gamma^{a\dagger} = \begin{cases} -\gamma^a & \text{if } a = 0 \\ \gamma^a & \text{if } a \neq 0 \end{cases} \quad (44)$$

$$\gamma^{a\dagger}\gamma^0 = -\gamma^0\gamma^a \quad (45)$$

$$\left(\frac{i}{2}\bar{\Psi}\gamma^a v_a^\mu D_\mu \Psi \right)^\dagger = \frac{-i}{2}(D_\mu \Psi)^\dagger \gamma^{a\dagger} v_a^\mu \gamma^{0\dagger} \Psi \quad (46)$$

$$= \frac{-i}{2}(D_\mu \Psi)^\dagger \gamma^{a\dagger} v_a^\mu (-\gamma^0) \Psi \quad (47)$$

$$= \frac{-i}{2}(D_\mu \Psi)^\dagger \gamma^0 \gamma^a v_a^\mu \Psi \quad (48)$$

$$= \frac{-i}{2}(\overline{D_\mu \Psi}) \gamma^a v_a^\mu \Psi \quad (49)$$

The previous Lagrangean density corresponds to the action,

$$S = \int dx^n \sqrt{g} \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2} \left(\bar{\Psi}\gamma^a v_a^\mu D_\mu \Psi - (\overline{D_\mu \Psi})\gamma^a v_a^\mu \Psi \right) + im\bar{\Psi}\Psi \right) \quad (50)$$

with $g = |\det(g_{\mu\nu})|$.

3 Coframe independance

I shall show that the Dirac equation in curved spacetimes is invariant under a local rotation of the coframe. The coframes θ_2 and θ_1 are related by,

$$\theta_2^a = \Lambda(x)^a_b \theta_1^b \quad (51)$$

With $\Lambda \in \text{SO}(3, 1)$. Λ can have different values at different events in space-time. The vector fields v_2 and v_1 are related by,

$$v_{2a} = \Lambda_a^b v_{1b} \quad (52)$$

That this works can be checked:

$$\theta_2^a(v_{2c}) = \Lambda^a_b \theta_1^b \Lambda_c^e v_{1e} \quad (53)$$

$$= \Lambda^a_b \Lambda_c^e \delta_e^b \quad (54)$$

$$= \Lambda^a_b \Lambda_c^b \quad (55)$$

$$= \delta_c^a \quad (56)$$

The last equation holds because Λ is in $\text{SO}(3, 1)$. Λ can be expressed as,

$$\Lambda^a_b = \left(\exp(k^c_d \mathfrak{J}_c^d) \right)^a_b \quad (57)$$

The notation \mathfrak{J}_c^d means a matrix with a 1 at row c column d and 0 at every other entry. The notation $()^a_b$ means row a column b of the matrix inside the parentheses. k is antisymmetric:

$$k_{ab} = -k_{ba} \quad (58)$$

I suppose there is a solution Ψ_1 with coframe θ_1 and look for another solution Ψ_2 with coframe θ_2 which has the same current:

$$j_1 = j_2 \quad (59)$$

$$-e \overline{\Psi}_1 \gamma^a \Psi_1 v_{1a} = -e \overline{\Psi}_2 \gamma^a \Psi_2 v_{2a} \quad (60)$$

$$-e \overline{\Psi}_1 \gamma^a \Psi_1 v_{1a} = -e \overline{\Psi}_2 \gamma^a \Psi_2 \Lambda_a^b v_{1b} \quad (61)$$

A solution is,

$$\Psi_2 = \Lambda_{\frac{1}{2}}(x) \Psi_1 \quad (62)$$

with,

$$\Lambda_{\frac{1}{2}}^{-1} \gamma^a \Lambda_{\frac{1}{2}} = \Lambda^a_b \gamma^b \quad (63)$$

$\Lambda_{\frac{1}{2}}$ can have different values at different events in spacetime. Later in this article, I shall give a method of constructing $\Lambda_{\frac{1}{2}}$ which leaves the current and Lagrangean invariant. The equation becomes,

$$-\overline{\Psi}_1 \gamma^a \Psi_1 v_{1a} = -\Psi_1^\dagger \Lambda_{\frac{1}{2}}^\dagger \gamma^0 \gamma^a \Lambda_{\frac{1}{2}} \Psi_1 \Lambda_a^b v_{1b} \quad (64)$$

I have,

$$\Lambda_{\frac{1}{2}}^\dagger \gamma^0 = \exp \left(\frac{1}{8} k_{ab} [\gamma^a, \gamma^b] \right)^\dagger \gamma^0 \quad (65)$$

$$= \left(1 + \frac{1}{8} k_{ab} [\gamma^a, \gamma^b] + \frac{1}{128} k_{ab} [\gamma^a, \gamma^b] k_{cd} [\gamma^c, \gamma^d] + \dots \right)^\dagger \gamma^0 \quad (66)$$

$$= \left(1 + \frac{1}{8} k_{ab} [\gamma^{b\dagger}, \gamma^{a\dagger}] + \frac{1}{128} k_{cd} [\gamma^{d\dagger}, \gamma^{c\dagger}] k_{ab} [\gamma^{b\dagger}, \gamma^{a\dagger}] + \dots \right) \gamma^0 \quad (67)$$

$$= \gamma^0 \left(1 + \frac{1}{8} k_{ab} [\gamma^b, \gamma^a] + \frac{1}{128} k_{cd} [\gamma^d, \gamma^c] k_{ab} [\gamma^b, \gamma^a] + \dots \right) \quad (68)$$

$$= \gamma^0 \left(1 - \frac{1}{8} k_{ab} [\gamma^a, \gamma^b] + \frac{1}{128} k_{ab} [\gamma^a, \gamma^b] k_{cd} [\gamma^c, \gamma^d] + \dots \right) \quad (69)$$

$$= \gamma^0 \Lambda_{\frac{1}{2}}^{-1} \quad (70)$$

then the equation becomes,

$$-\overline{\Psi}_1 \gamma^a \Psi_1 v_{1a} = -\Psi_1^\dagger \gamma^0 \Lambda_{\frac{1}{2}}^{-1} \gamma^a \Lambda_{\frac{1}{2}} \Psi_1 \Lambda_a^b v_{1b} \quad (71)$$

$$= -\overline{\Psi}_1 \Lambda_c^a \gamma^c \Psi_1 \Lambda_a^b v_{1b} \quad (72)$$

$$= -\overline{\Psi}_1 \gamma^a \Psi_1 v_{1a} \quad (73)$$

The last equation holds because Λ is in $\text{SO}(3, 1)$.

Coframe transformations can be composited. Given two coframe transformations $(\Lambda_1, \Lambda_{\frac{1}{2}1}) : (\theta_1, \Psi_1) \rightarrow (\theta_2, \Psi_2)$ and $(\Lambda_2, \Lambda_{\frac{1}{2}2}) : (\theta_2, \Psi_2) \rightarrow (\theta_3, \Psi_3)$, there is a composite coframe transformation $(\Lambda_3, \Lambda_{\frac{1}{2}3}) : (\theta_1, \Psi_1) \rightarrow (\theta_3, \Psi_3)$

which makes this diagram commute:

$$\begin{array}{ccc}
 (\theta_1, \Psi_1) & \xrightarrow{(\Lambda_1, \Lambda_{\frac{1}{2}1})} & (\theta_2, \Psi_2) \\
 & \searrow (\Lambda_3, \Lambda_{\frac{1}{2}3}) & \downarrow (\Lambda_2, \Lambda_{\frac{1}{2}2}) \\
 & & (\theta_3, \Psi_3)
 \end{array}$$

It is,

$$\Lambda_3^a{}_b = \Lambda_2^a{}_c \Lambda_1^c{}_b \quad (74)$$

$$\Lambda_{\frac{1}{2}3} = \Lambda_{\frac{1}{2}2} \Lambda_{\frac{1}{2}1} \quad (75)$$

Because the end values of the fields θ and Ψ are the same for whichever path is taken in the previous commutative diagram, and θ and Ψ are the only fields which enter the current and the Lagrangean which are affected by a coframe transformation, it follows that $(\Lambda_3, \Lambda_{\frac{1}{2}3})$ leave the current and the Lagrangean invariant if $(\Lambda_1, \Lambda_{\frac{1}{2}1})$ and $(\Lambda_2, \Lambda_{\frac{1}{2}2})$ leave the current and the Lagrangean invariant.

An arbitrary coframe transformation $\Lambda_{\text{arbitrary}}$ can be decomposed into Euler angles:

$$\Lambda_{\text{arbitrary}} = \Lambda_6 \Lambda_5 \Lambda_4 \Lambda_3 \Lambda_2 \Lambda_1 \quad (76)$$

The coframe transformations Λ_1 through Λ_6 have all but one of the k_{ab} equal to 0 everywhere in spacetime. Without loss of generality, I can assume that for Λ all but one of the k_{ab} are zero everywhere in spacetime and that,

$$\Lambda_{\frac{1}{2}} = \exp \left(\frac{1}{2} k_{ab} S^{ab} \right) \quad (77)$$

It can be checked that $\Lambda_{\frac{1}{2}}$ as defined in equation (77) follows equation (63): I denote r and s the fixed indices of the Euler angle with $r < s$. If k_{rs} is a spatial rotation, then,

$$\exp \left(\frac{1}{2} k_{ab} S^{ab} \right) = \exp \left(\frac{1}{2} k_{rs} \gamma^r \gamma^s \right) \quad (78)$$

$$= \cos \left(\frac{k_{rs}}{2} \right) + \sin \left(\frac{k_{rs}}{2} \right) \gamma^r \gamma^s \quad (79)$$

$$\begin{aligned}
\Lambda_{\frac{1}{2}}^{-1} \gamma^a \Lambda_{\frac{1}{2}} &= \\
&= \left(\cos \left(\frac{k_{rs}}{2} \right) - \sin \left(\frac{k_{rs}}{2} \right) \gamma^r \gamma^s \right) \gamma^a \times \\
&\quad \times \left(\cos \left(\frac{k_{rs}}{2} \right) + \sin \left(\frac{k_{rs}}{2} \right) \gamma^r \gamma^s \right) \tag{80}
\end{aligned}$$

$$= \cos \left(\frac{k_{rs}}{2} \right)^2 \gamma^a + \cos \left(\frac{k_{rs}}{2} \right) \sin \left(\frac{k_{rs}}{2} \right) (\gamma^a \gamma^r \gamma^s - \gamma^r \gamma^s \gamma^a) - \tag{81}$$

$$- \sin \left(\frac{k_{rs}}{2} \right)^2 \gamma^r \gamma^s \gamma^a \gamma^r \gamma^s \tag{82}$$

$$= \cos \left(\frac{k_{rs}}{2} \right)^2 \gamma^a + \cos \left(\frac{k_{rs}}{2} \right) \sin \left(\frac{k_{rs}}{2} \right) (2\eta^{ar} \gamma^s - 2\eta^{as} \gamma^r) - \tag{83}$$

$$- \sin \left(\frac{k_{rs}}{2} \right)^2 (\gamma^r \eta^{ar} \eta^{ss} + \gamma^s \eta^{as} \eta^{rr} - \gamma^a \eta^{rr} \eta^{ss} (1 - \delta_r^a)(1 - \delta_s^a)) \tag{84}$$

If $a = r$,

$$= \left(\cos \left(\frac{k_{rs}}{2} \right)^2 - \sin \left(\frac{k_{rs}}{2} \right)^2 \right) \gamma^a + 2 \cos \left(\frac{k_{rs}}{2} \right) \sin \left(\frac{k_{rs}}{2} \right) \gamma^s \tag{85}$$

$$= \cos(k_{rs}) \gamma^a + \sin(k_{rs}) \gamma^s \tag{86}$$

$$\Lambda^a_b \gamma^b = \left(\exp(k_{rs}(\mathfrak{J}^{rs} - \mathfrak{J}^{sr})) \right)^a_b \gamma^b \tag{87}$$

$$= \left((\cos(k_{rs}) - 1)(\mathfrak{J}_r^r + \mathfrak{J}_s^s) + \delta_d^c \mathfrak{J}_c^d + \sin(k_{rs})(\mathfrak{J}^{rs} - \mathfrak{J}^{sr}) \right)^a_b \gamma^b \tag{88}$$

If $a = r$,

$$= \cos(k_{rs}) \gamma^a + \sin(k_{rs}) \gamma^s \tag{89}$$

The case $a = s$ is similar by antisymmetry. If k_{rs} is a boost, the proof is similar, *mutatis mutandis*.

An algorithm for assigning values to $\Lambda_{\frac{1}{2}\text{arbitrary}}$ is to decompose $\Lambda_{\text{arbitrary}}$ into Euler angles as in equation (76), then assigning values to $\Lambda_{\frac{1}{2}1}$ through $\Lambda_{\frac{1}{2}6}$ using equation (77), then calculating,

$$\Lambda_{\frac{1}{2}\text{arbitrary}} = \Lambda_{\frac{1}{2}6} \Lambda_{\frac{1}{2}5} \Lambda_{\frac{1}{2}4} \Lambda_{\frac{1}{2}3} \Lambda_{\frac{1}{2}2} \Lambda_{\frac{1}{2}1} \tag{90}$$

If the Lagrangean density is independent of the local orientation of the coframe, then Ψ_2 is a solution of the curved spacetimes Dirac equation with

coframe θ_2 : If

$$\mathfrak{L}[\theta_2, \Psi_2] = \mathfrak{L}[\theta_1, \Psi_1] \quad (91)$$

then,

$$\mathfrak{L}[\theta_2, \Psi_2 + \delta\Psi_2] - \mathfrak{L}[\theta_2, \Psi_2] = \mathfrak{L}[\theta_1, \Psi_1 + \Lambda_{\frac{1}{2}}^{-1}\delta\Psi_2] - \mathfrak{L}[\theta_1, \Psi_1] \quad (92)$$

If Ψ_1 follows the wave equation with coframe θ_1 , then the right side of the previous equation is a 4-divergence, and therefore Ψ_2 follows the wave equation with coframe θ_2 .

Following is a calculation that shows equation (91):

$$\begin{aligned} \mathfrak{L}[\theta_2, \Psi_2] &= -\frac{1}{2}dA \wedge *dA - e\overline{\Psi}_2\gamma^a A \wedge *\theta_{2a}\Psi_2 + \\ &\quad + i\overline{\Psi}_2 \left(\gamma^a d\Psi_2 \wedge *\theta_{2a} - \frac{1}{4}\gamma^a\gamma^b\gamma^c\omega_{2bc} \wedge *\theta_{2a}\Psi_2 + m\Psi_2 *1 \right) \end{aligned} \quad (93)$$

$$\begin{aligned} &= -\frac{1}{2}dA \wedge *dA - e\overline{\Psi}_2\gamma^a A(v_2^a)\Psi_2 *1 + \\ &\quad + i\overline{\Psi}_2 \left(\gamma^a d\Psi_2(v_{2a}) - \frac{1}{4}\gamma^a\gamma^b\gamma^c\omega_{2bc}(v_{2a})\Psi_2 + m\Psi_2 \right) *1 \end{aligned} \quad (94)$$

Because θ_1 and θ_2 represent the same metric, the Hodge dual is the same using either coframe.

$$\begin{aligned} \mathfrak{L}[\theta_2, \Psi_2] &= -\frac{1}{2}dA \wedge *dA - e\overline{\Psi}_1\Lambda_{\frac{1}{2}}^{-1}\gamma^a A(v_2^a)\Lambda_{\frac{1}{2}}\Psi_1 *1 + \\ &\quad + i\overline{\Psi}_1\Lambda_{\frac{1}{2}}^{-1} \left(\gamma^a \Lambda_{\frac{1}{2}} d\Psi_1(v_{2a}) + \gamma^a \left(v_{2a}\Lambda_{\frac{1}{2}} \right) \Psi_1 - \frac{1}{4}\gamma^a\gamma^b\gamma^c\omega_{2bc}(v_{2a}) + \right. \\ &\quad \left. + m\Lambda_{\frac{1}{2}}\Psi_1 \right) *1 \end{aligned} \quad (95)$$

The second term on the right of equation (95) is,

$$-e\overline{\Psi}_1\Lambda_{\frac{1}{2}}^{-1}\gamma^a A(v_2^a)\Lambda_{\frac{1}{2}}\Psi_1 *1 = -e\overline{\Psi}_1\Lambda_{\frac{1}{2}}^{-1}\gamma^a\Lambda_{\frac{1}{2}}A(v_{2a})\Psi_1 *1 \quad (96)$$

$$= -e\overline{\Psi}_1\Lambda^a{}_b\gamma^b A(v_{2a})\Psi_1 *1 \quad (97)$$

$$= -e\overline{\Psi}_1\gamma^a A(v_{1a})\Psi_1 *1 \quad (98)$$

I apply,

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^a\Lambda_{\frac{1}{2}} = \Lambda^a{}_b\gamma^b \quad (99)$$

to the third term on the right of equation (95):

$$i\overline{\Psi}_1\Lambda_{\frac{1}{2}}^{-1}\gamma^a\Lambda_{\frac{1}{2}}(v_{2a}\Psi_1)*1=i\overline{\Psi}_1\Lambda^a{}_b\gamma^b(v_{2a}\Psi_1)*1 \quad (100)$$

$$=i\overline{\Psi}_1\Lambda^a{}_b\gamma^b\Lambda_a{}^c(v_{1c}\Psi_1)*1 \quad (101)$$

$$=i\overline{\Psi}_1\gamma^a(v_{1a}\Psi_1)*1 \quad (102)$$

The fourth term on the right of equation (95) is,

$$i\overline{\Psi}_1\Lambda_{\frac{1}{2}}^{-1}\gamma^a\left(v_{2a}\Lambda_{\frac{1}{2}}\right)\Psi_1*1=i\overline{\Psi}_1\Lambda_{\frac{1}{2}}^{-1}\gamma^a\left(v_{2a}\exp\left(\frac{1}{2}k_{bc}S^{bc}\right)\right)\Psi_1*1 \quad (103)$$

Because all but one of the k_{bc} are 0 everywhere in spacetime, $\Lambda_{\frac{1}{2}}$ commutes with its derivative:

$$=\frac{i}{2}\overline{\Psi}_1\Lambda_{\frac{1}{2}}^{-1}\gamma^a\Lambda_{\frac{1}{2}}S^{bc}(v_{2a}k_{bc})\Psi_1*1 \quad (104)$$

$$=\frac{i}{2}\overline{\Psi}_1\Lambda^a{}_d\gamma^dS^{bc}(v_{2a}k_{bc})\Psi_1*1 \quad (105)$$

$$=\frac{i}{2}\overline{\Psi}_1\gamma^aS^{bc}(v_{1a}k_{bc})\Psi_1*1 \quad (106)$$

The fifth term on the right of equation (95) is,

$$-i\overline{\Psi}_1\Lambda_{\frac{1}{2}}^{-1}\frac{1}{4}\gamma^a\gamma^b\gamma^c\omega_{2bc}(v_{2a})\Lambda_{\frac{1}{2}}\Psi_1*1 \quad (107)$$

with,

$$\omega_{2bc}(v_{2a})=\frac{1}{2}\left(d\theta_{2a}(v_{2b},v_{2c})+d\theta_{2b}(v_{2a},v_{2c})-d\theta_{2c}(v_{2a},v_{2b})\right) \quad (108)$$

Raising the index a , the first term inside the parentheses on the right of the previous equation is,

$$d\theta_2{}^a(v_{2b},v_{2c})=d(\Lambda^a{}_f\theta_1{}^f)(v_{2b},v_{2c}) \quad (109)$$

$$=d\left((\exp(k^d{}_e\mathfrak{J}_d{}^e))^a{}_f\theta_1{}^f\right)(v_{2b},v_{2c}) \quad (110)$$

$$=\left(v_{1g}(\exp(k^d{}_e\mathfrak{J}_d{}^e))^a{}_f\right)(\theta_1{}^g\wedge\theta_1{}^f)(v_{2b},v_{2c})+\\ +\Lambda^a{}_fd\theta_1{}^f(v_{2b},v_{2c}) \quad (111)$$

The first term on the right of the previous equation is,

$$\left(v_{1g}(\exp(k^d{}_e\mathfrak{J}_d{}^e))^a{}_f\right)(\theta_1{}^g\wedge\theta_1{}^f)(v_{2b},v_{2c})=\\ =\left(v_{1g}\exp(k^d{}_e\mathfrak{J}_d{}^e)\right)^a{}_f(\theta_1{}^g\wedge\theta_1{}^f)(v_{2b},v_{2c}) \quad (112)$$

Because all but one of the k_{de} are 0 everywhere in spacetime, Λ commutes with its derivative. A typical term of the power series expansion of $\Lambda = \exp(k_{de}\mathfrak{J}^{de})$ is,

$$\frac{1}{n!}(k_{rs})^n(\mathfrak{J}^{rs} - \mathfrak{J}^{sr})^n \quad (113)$$

Its derivative is,

$$\begin{aligned} \frac{1}{n!}n(k_{rs})^{n-1}(\mathfrak{J}^{rs} - \mathfrak{J}^{sr})^n v_{1g}k_{rs} &= \\ &= \frac{1}{(n-1)!}(k_{rs})^{n-1}(\mathfrak{J}^{rs} - \mathfrak{J}^{sr})^{n-1}(\mathfrak{J}^{rs} - \mathfrak{J}^{sr})v_{1g}k_{rs} \end{aligned} \quad (114)$$

So the derivative of Λ is,

$$v_{1g}\Lambda = \Lambda\mathfrak{J}^{de}v_{1g}k_{de} \quad (115)$$

So the right side of equation (112) becomes,

$$\begin{aligned} &\left(\exp(k_e^d\mathfrak{J}_d^e)\mathfrak{J}_h^i v_{1g}k_i^h\right)_f^a (\theta_1^g \wedge \theta_1^f)(v_{2b}, v_{2c}) \\ &= \left(\exp(k_e^d\mathfrak{J}_d^e)\mathfrak{J}_h^i\right)_f^a (v_{1g}k_i^h)(\theta_1^g \wedge \theta_1^f)(v_{2b}, v_{2c}) \end{aligned} \quad (116)$$

$$= \Lambda_h^a \delta_f^i (v_{1g}k_i^h)(\theta_1^g \wedge \theta_1^f)(v_{2b}, v_{2c}) \quad (117)$$

$$= \Lambda_h^a (v_{1g}k_f^h)(\theta_1^g \wedge \theta_1^f)(v_{2b}, v_{2c}) \quad (118)$$

$$= \Lambda_h^a (v_{1g}k_f^h) \left(\theta_1^g(v_{2b})\theta_1^f(v_{2c}) - \theta_1^g(v_{2c})\theta_1^f(v_{2b}) \right) \quad (119)$$

$$= \Lambda_h^a (v_{1g}k_f^h) \left(\theta_1^g(\Lambda_b^j v_{1j})\theta_1^f(\Lambda_c^k v_{1k}) - \theta_1^g(\Lambda_c^k v_{1k})\theta_1^f(\Lambda_b^j v_{1j}) \right) \quad (120)$$

$$= \Lambda_h^a (v_{1g}k_f^h) \left((\Lambda_b^j \delta_j^g)(\Lambda_c^k \delta_k^f) - (\Lambda_c^k \delta_k^g)(\Lambda_b^j \delta_j^f) \right) \quad (121)$$

$$= \Lambda_h^a \Lambda_b^g \Lambda_c^f \left((v_{1g}k_f^h) - (v_{1f}k_g^h) \right) \quad (122)$$

$$= \Lambda_f^a \Lambda_b^g \Lambda_c^h \left((v_{1g}k_h^f) - (v_{1h}k_g^f) \right) \quad (123)$$

The second term on the right of equation (111) is,

$$\Lambda_f^a d\theta_1^f(v_{2b}, v_{2c}) = \Lambda_f^a \Lambda_b^g \Lambda_c^h d\theta_1^f(v_{1g}, v_{1h}) \quad (124)$$

Lowering the index a , I get,

$$\begin{aligned} d\theta_{2a}(v_{2b}, v_{2c}) &= \Lambda_a^f \Lambda_b^g \Lambda_c^h d\theta_{1f}(v_{1g}, v_{1h}) + \\ &+ \Lambda_a^f \Lambda_b^g \Lambda_c^h (v_{1g}k_{fh} - v_{1h}k_{fg}) \end{aligned} \quad (125)$$

Similarly,

$$\begin{aligned} \omega_{2bc}(v_{2a}) &= \Lambda_a^f \Lambda_b^g \Lambda_c^h \left(\omega_{1gh}(v_{1f}) + \frac{1}{2} \left(v_{1g}k_{fh} - v_{1h}k_{fg} + \right. \right. \\ &\quad \left. \left. + v_{1f}k_{gh} - v_{1h}k_{gf} - v_{1f}k_{hg} + v_{1g}k_{hf} \right) \right) \end{aligned} \quad (126)$$

and because k is antisymmetric,

$$\omega_{2bc}(v_{2a}) = \Lambda_a^f \Lambda_b^g \Lambda_c^h \left(\omega_{1gh}(v_{1f}) + v_{1f}k_{gh} \right) \quad (127)$$

The fifth term on the right of equation (95) becomes,

$$\begin{aligned} -\frac{1}{4} \overline{\Psi_1} \Lambda_{\frac{1}{2}}^{-1} \gamma^a \gamma^b \gamma^c \omega_{2bc}(v_{2a}) \Lambda_{\frac{1}{2}} \Psi_1 &= \\ = -\frac{1}{4} \overline{\Psi_1} \Lambda_{\frac{1}{2}}^{-1} \gamma^a \gamma^b \gamma^c \Lambda_{\frac{1}{2}} \Lambda_a^f \Lambda_b^g \Lambda_c^h \left(\omega_{1gh}(v_{1f}) + v_{1f}k_{gh} \right) \Psi_1 \end{aligned} \quad (128)$$

Using that,

$$\Lambda_a^b \gamma^a = \Lambda_{\frac{1}{2}} \gamma^b \Lambda_{\frac{1}{2}}^{-1} \quad (129)$$

The fifth term on the right of equation (95) becomes,

$$\begin{aligned} -\frac{i}{4} \overline{\Psi_1} \Lambda_{\frac{1}{2}}^{-1} \gamma^a \gamma^b \gamma^c \omega_{2bc}(v_{2a}) \Lambda_{\frac{1}{2}} \Psi_1 * 1 &= \\ = -\frac{i}{4} \overline{\Psi_1} \Lambda_{\frac{1}{2}}^{-1} \Lambda_{\frac{1}{2}} \gamma^a \Lambda_{\frac{1}{2}}^{-1} \Lambda_{\frac{1}{2}} \gamma^b \Lambda_{\frac{1}{2}}^{-1} \Lambda_{\frac{1}{2}} \gamma^c \Lambda_{\frac{1}{2}}^{-1} \Lambda_{\frac{1}{2}} &\times \\ \times \left(\omega_{1bc}(v_{1a}) + v_{1a}k_{bc} \right) \Psi_1 * 1 \end{aligned} \quad (130)$$

$$= -\frac{i}{4} \overline{\Psi_1} \gamma^a \gamma^b \gamma^c \left(\omega_{1bc}(v_{1a}) + v_{1a}k_{bc} \right) \Psi_1 * 1 \quad (131)$$

Gathering terms, equation (95) becomes,

$$\begin{aligned}\mathfrak{L}[\theta_2, \Psi_2] &= -\frac{1}{2}dA \wedge *dA - e\overline{\Psi}_1\gamma^a A(v_{1a})\Psi_1 *1 + \\ &\quad + i\overline{\Psi}_1\left(\gamma^a v_{1a} + \frac{1}{2}\gamma^a S^{bc}v_{1a}k_{bc} - \right. \\ &\quad \left. - \frac{1}{4}\gamma^a\gamma^b\gamma^c(\omega_{1bc}(v_{1a}) + v_{1a}k_{bc}) + m\right)\Psi_1 *1\end{aligned}\quad (132)$$

$$\begin{aligned}&= -\frac{1}{2}dA \wedge *dA - e\overline{\Psi}_1\gamma^a A(v_{1a})\Psi_1 *1 + \\ &\quad + i\overline{\Psi}_1\left(\gamma^a v_{1a} - \frac{1}{4}\gamma^a\gamma^b\gamma^c\omega_{1bc}(v_{1a}) + m\right)\Psi_1 *1\end{aligned}\quad (133)$$

$$= \mathfrak{L}[\theta_1, \Psi_1]\quad (134)$$

This shows that the Lagrangean density is independent of the local orientation of the coframe.

4 Conservation of current

I now calculate,

$$d\Psi \wedge *\theta_0 = d\Psi \wedge \theta^1 \wedge \theta^2 \wedge \theta^3\quad (135)$$

$$= d\Psi(v_0)\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3\quad (136)$$

$$= d\Psi(v_0) *1\quad (137)$$

Therefore,

$$*(d\Psi \wedge *\theta_0) = *(d\Psi(v_0) *1)\quad (138)$$

$$= d\Psi(v_0) **1\quad (139)$$

$$= -d\Psi(v_0)\quad (140)$$

Generally,

$$*(d\Psi \wedge *\theta_a) = -d\Psi(v_a)\quad (141)$$

So,

$$v_a\Psi = - * (d\Psi \wedge *\theta_a)\quad (142)$$

Substituting $v_a \Psi = -*(d\Psi \wedge * \theta_a)$ in the Dirac equation in curved spacetimes, I get,

$$-\gamma^a * (d\Psi \wedge * \theta_a) + ie\gamma^a A(v_a)\Psi - \frac{1}{4}\gamma^a \gamma^b \gamma^c \omega_{bc}(v_a)\Psi + m\Psi = 0 \quad (143)$$

Its Hermitean conjugate is,

$$- * (d\Psi^\dagger \wedge * \theta_a) \gamma^{a\dagger} - ie\Psi^\dagger \gamma^{a\dagger} A(v_a) - \frac{1}{4}\omega_{bc}(v_a)\Psi^\dagger \gamma^{c\dagger} \gamma^{b\dagger} \gamma^{a\dagger} + m\Psi^\dagger = 0 \quad (144)$$

Multiplying by γ^0 on the right,

$$- * (d\Psi^\dagger \wedge * \theta_a) \gamma^{a\dagger} \gamma^0 - ie\Psi^\dagger \gamma^{a\dagger} \gamma^0 A(v_a) - \frac{1}{4}\omega_{bc}(v_a)\Psi^\dagger \gamma^{c\dagger} \gamma^{b\dagger} \gamma^{a\dagger} \gamma^0 + m\Psi^\dagger \gamma^0 = 0 \quad (145)$$

$$* (d\Psi^\dagger \wedge * \theta_a) \gamma^0 \gamma^a + ie\Psi^\dagger \gamma^0 \gamma^a A(v_a) + \frac{1}{4}\omega_{bc}(v_a)\Psi^\dagger \gamma^0 \gamma^c \gamma^b \gamma^a + m\Psi^\dagger \gamma^0 = 0 \quad (146)$$

$$* (d\bar{\Psi} \wedge * \theta_a) \gamma^a + ie\bar{\Psi} \gamma^a A(v_a) + \frac{1}{4}\omega_{bc}(v_a)\bar{\Psi} \gamma^c \gamma^b \gamma^a + m\bar{\Psi} = 0 \quad (147)$$

The current is,

$$j = -e\bar{\Psi} \gamma^a \Psi * \theta_a \quad (148)$$

Conservation of current means,

$$*dj = 0 \quad (149)$$

$$-e * (d\bar{\Psi} \wedge * \theta_a) \gamma^a \Psi - e\bar{\Psi} \gamma^a * (d\Psi \wedge * \theta_a) - e\bar{\Psi} \gamma^a \Psi * d* \theta_a = 0 \quad (150)$$

Using equations (143) and (147),

$$\frac{e}{4}\omega_{bc}(v_a)\bar{\Psi} (\gamma^a \gamma^b \gamma^c + \gamma^c \gamma^b \gamma^a) \Psi - e\bar{\Psi} \gamma^a \Psi * d* \theta_a = 0 \quad (151)$$

The products of three different gammas cancel leaving,

$$\frac{e}{4}\bar{\Psi} (4\gamma^c \omega^a_c(v_a)) \Psi - e\bar{\Psi} \gamma^a \Psi * d* \theta_a = 0 \quad (152)$$

$$e\bar{\Psi} (\gamma^b \omega^a_b(v_a)) \Psi - e\bar{\Psi} \gamma^a \Psi * d* \theta_a = 0 \quad (153)$$

$$\omega^a_b(v_a) = -d\theta^a(v_b, v_a) \quad (154)$$

$$-e\bar{\Psi} \gamma^b \Psi d\theta^a(v_b, v_a) - e\bar{\Psi} \gamma^a \Psi * d* \theta_a = 0 \quad (155)$$

For the last term,

$$*\theta_0 = \theta^1 \wedge \theta^2 \wedge \theta^3 \quad (156)$$

$$d*\theta_0 = d\theta^1 \wedge \theta^2 \wedge \theta^3 - \theta^1 \wedge d\theta^2 \wedge \theta^3 + \theta^1 \wedge \theta^2 \wedge d\theta^3 \quad (157)$$

$$*(d\theta^1 \wedge \theta^2 \wedge \theta^3) = *(d\theta^1(v_0, v_1)\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3) \quad (158)$$

$$= *(d\theta^1(v_0, v_1) * 1) \quad (159)$$

$$= d\theta^1(v_0, v_1) **1 \quad (160)$$

$$= -d\theta^1(v_0, v_1) \quad (161)$$

Similarly,

$$*d*\theta_a = -d\theta^b(v_a, v_b) \quad (162)$$

And the equation for the conservation of current becomes,

$$*dj = -e\bar{\Psi}\gamma^a\Psi d\theta^b(v_a, v_b) + e\bar{\Psi}\gamma^a\Psi d\theta^b(v_a, v_b) \quad (163)$$

$$= 0 \quad (164)$$

5 Alternative formulation

The term,

$$\omega_{bc}(v_a) \quad (165)$$

can be written in an alternative formulation:

$$\omega_{bc}(v_a) = \frac{1}{2} \left(d\theta_a(v_b, v_c) + d\theta_b(v_a, v_c) - d\theta_c(v_a, v_b) \right) \quad (166)$$

$$= \frac{1}{2} \left(d\theta_a(v_b^\nu \partial_\nu, v_c^\rho \partial_\rho) + d\theta_b(v_a^\mu \partial_\mu, v_c^\rho \partial_\rho) - d\theta_c(v_a^\mu \partial_\mu, v_b^\nu \partial_\nu) \right) \quad (167)$$

$$= \frac{1}{2} \left(v_b^\nu v_c^\rho (\partial_\nu \theta_{a\rho} - \partial_\rho \theta_{a\nu}) + v_a^\mu v_c^\rho (\partial_\mu \theta_{b\rho} - \partial_\rho \theta_{b\mu}) + \right. \\ \left. + v_a^\mu v_b^\nu (-\partial_\mu \theta_{c\nu} + \partial_\nu \theta_{c\mu}) \right) \quad (168)$$

Writing the Christoffel gamma as,

$$\Gamma_{\rho\nu\mu} = \frac{1}{2} \left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\nu\mu} \right) \quad (169)$$

and using that,

$$g_{\mu\nu} = \eta_{ab} \theta^a{}_\mu \theta^b{}_\nu \quad (170)$$

the Christoffel gamma becomes,

$$\Gamma_{\rho\nu\mu} = \frac{1}{2} \left(\eta_{de} \partial_\mu (\theta^d{}_\rho \theta^e{}_\nu) + \eta_{de} \partial_\nu (\theta^d{}_\rho \theta^e{}_\mu) - \eta_{de} \partial_\rho (\theta^d{}_\nu \theta^e{}_\mu) \right) \quad (171)$$

$$= \frac{1}{2} \left(\theta^d{}_\nu \partial_\mu \theta_{d\rho} + \theta^d{}_\rho \partial_\mu \theta_{d\nu} + \theta^d{}_\mu \partial_\nu \theta_{d\rho} + \theta^d{}_\rho \partial_\nu \theta_{d\mu} - \theta^d{}_\mu \partial_\rho \theta_{d\nu} - \theta^d{}_\nu \partial_\rho \theta_{d\mu} \right) \quad (172)$$

$$\begin{aligned} v_a{}^\mu v_b{}^\nu v_c{}^\rho \Gamma_{\rho\nu\mu} &= \frac{1}{2} \left(v_a{}^\mu v_c{}^\rho (\partial_\mu \theta_{b\rho} - \partial_\rho \theta_{b\mu}) + v_a{}^\mu v_b{}^\nu (\partial_\mu \theta_{c\nu} + \partial_\nu \theta_{c\mu}) + \right. \\ &\quad \left. + v_b{}^\nu v_c{}^\rho (\partial_\nu \theta_{a\rho} - \partial_\rho \theta_{a\nu}) \right) \end{aligned} \quad (173)$$

Inserting into the equation for ω , I get,

$$\omega_{bc}(v_a) = v_a{}^\mu v_b{}^\nu \theta_{c\rho} \Gamma_{\nu\mu}^\rho - v_a{}^\mu v_b{}^\nu \partial_\mu \theta_{c\nu} \quad (174)$$

An alternative formulation for the Dirac equation in curved spacetimes is,

$$\left(\gamma^a v_a{}^\mu \partial_\mu + ie \gamma^a A(v_a) - \frac{1}{2} \gamma^a S^{bc} v_a{}^\mu v_b{}^\nu (\theta_{c\rho} \Gamma_{\nu\mu}^\rho - \partial_\mu \theta_{c\nu}) + m \right) \Psi = 0 \quad (175)$$

Appendix 1

ω has the same values as the rotational part of the torsion-free Cartan connection $(\theta, \omega) : T(\mathcal{M}^4) \rightarrow \mathfrak{euc}(3, 1)$ because ω is antisymmetric and follows the equation,

$$d\theta^a - \omega^a{}_b \wedge \theta^b = 0 \quad (176)$$

To prove equation (176):

$$\begin{aligned} d\theta^a(v_c, v_d) - (\omega^a{}_b \wedge \theta^b)(v_c, v_d) &= \\ &= d\theta^a(v_c, v_d) - \omega^a{}_b(v_c) \theta^b(v_d) + \omega^a{}_b(v_d) \theta^b(v_c) \end{aligned} \quad (177)$$

$$= d\theta^a(v_c, v_d) - \omega^a{}_d(v_c) + \omega^a{}_c(v_d) \quad (178)$$

$$\begin{aligned} &= d\theta^a(v_c, v_d) - \frac{1}{2} \left(d\theta_c(v^a, v_d) + d\theta^a(v_c, v_d) - d\theta_d(v_c, v^a) \right) + \\ &\quad + \frac{1}{2} \left(d\theta_d(v^a, v_c) + d\theta^a(v_d, v_c) - d\theta_c(v_d, v^a) \right) \end{aligned} \quad (179)$$

$$= d\theta^a(v_c, v_d) - d\theta^a(v_c, v_d) \quad (180)$$

$$= 0 \quad (181)$$

To prove that equation (176) is the formula for the torsion: The formula for the curvature of a Cartan connection ξ is [5],

$$d\xi + \frac{1}{2}[\xi, \xi] \quad (182)$$

with,

$$[\xi, \chi](U, V) = [\xi(U), \chi(V)] - [\xi(V), \chi(U)] \quad (183)$$

with the commutator evaluated in a Lie algebra. For the Cartan connection $(\theta, \omega) : T(\mathcal{M}^4) \rightarrow \mathfrak{euc}(3, 1)$, the torsion is the translational part of the curvature:

$$d\theta + \frac{1}{2}[\theta, \omega] + \frac{1}{2}[\omega, \theta] \quad (184)$$

To simplify:

$$(d\theta + \frac{1}{2}[\theta, \omega] + \frac{1}{2}[\omega, \theta])(U, V) = \quad (185)$$

$$= d\theta(U, V) + \frac{1}{2}[\theta(U), \omega(V)] - \frac{1}{2}[\theta(V), \omega(U)] + \frac{1}{2}[\omega(U), \theta(V)] - \frac{1}{2}[\omega(V), \theta(U)] \quad (186)$$

$$= d\theta(U, V) + [\omega(U), \theta(V)] - [\omega(V), \theta(U)] \quad (187)$$

$$= (d\theta + [\omega, \theta])(U, V) \quad (188)$$

so the torsion is,

$$d\theta + [\omega, \theta] \quad (189)$$

Some of the generators of $\text{Euc}(3, 1)$ are,

$$J^{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad t^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (190)$$

They follow the commutation relations,

$$[J^{ab}, t^c] = -\eta^{bc}t^a + \eta^{ac}t^b \quad (191)$$

In this appendix, the values of ω are combinations of J matrices:

$$\omega = \frac{1}{2} J^{ab} \omega_{ab} \quad (192)$$

The rotation generators are duplicated (for example, there is J^{21} in addition to J^{12}), hence the factor $\frac{1}{2}$. Formula (189) applied to the vectors U and V gives,

$$(\mathrm{d}\theta + [\omega, \theta])^a(U, V) = \mathrm{d}\theta^a(U, V) + [\omega(U), \theta(V)]^a - [\omega(V), \theta(U)]^a \quad (193)$$

$$= (\mathrm{d}\theta^a - \frac{1}{2} \omega^a_b \wedge \theta^b + \frac{1}{2} \omega_b^a \wedge \theta^b)(U, V) \quad (194)$$

$$= (\mathrm{d}\theta^a - \omega^a_b \wedge \theta^b)(U, V) \quad (195)$$

So the torsion is,

$$\mathrm{d}\theta^a - \omega^a_b \wedge \theta^b \quad (196)$$

Appendix 2

In this appendix, I calculate the stress-energy tensor of the electromagnetic and Dirac fields in teleparallel gravity. The result is asymmetric, which is a known inconsistency [7] [8] in a version of the teleparallel equivalent of general relativity [6] in the presence of a Dirac field.

The Lagrangean density is,

$$\mathfrak{L} = \mathfrak{L}_{\text{em}} + \mathfrak{L}_{\text{int}} + \mathfrak{L}_{\text{Dirac}} \quad (197)$$

$$\begin{aligned} &= -\frac{1}{2} \mathrm{d}A \wedge * \mathrm{d}A - e \bar{\Psi} \gamma^a A \wedge * \theta_a \Psi + \\ &\quad + i \bar{\Psi} \left(\gamma^a \mathrm{d}\Psi \wedge * \theta_a - \frac{1}{4} \gamma^a \gamma^b \gamma^c \omega_{bc} \wedge * \theta_a \Psi + m \Psi * 1 \right) \end{aligned} \quad (198)$$

To calculate the stress-energy tensor in teleparallel gravity, I set up a small variation of the coframe which vanishes on the region boundary. The variation in the Lagrangean density is,

$$\begin{aligned} \delta \mathfrak{L} = & -\frac{1}{2} \delta(\mathrm{d}A \wedge * \mathrm{d}A) - e \bar{\Psi} \gamma^a \Psi A \wedge \delta * \theta_a + i \bar{\Psi} \left(\gamma^a \mathrm{d}\Psi \wedge \delta * \theta_a - \right. \\ & \left. - \frac{1}{4} \gamma^a \gamma^b \gamma^c \left(\delta(\omega_{bc}(v_a) * 1) \right) \Psi + m \Psi \delta * 1 \right) \end{aligned} \quad (199)$$

The first term on the right of equation (199) is,

$$-\frac{1}{2}\delta(\mathrm{d}A \wedge *\mathrm{d}A) \quad (200)$$

Denoting,

$$F_{ab} = \mathrm{d}A(v_a, v_b) \quad (201)$$

the first term can be written as,

$$-\frac{1}{2}\delta(\mathrm{d}A \wedge *\mathrm{d}A) = -\frac{1}{4}\delta(F_{ab}\eta^{ac}\eta^{bd}F_{cd}*1) \quad (202)$$

$$= -\frac{1}{2}F^{ab}\delta(F_{ab}*1) + \frac{1}{4}F^{ab}F_{ab}\delta*1 \quad (203)$$

$$= -\frac{1}{2}F^{ab}\delta\left(\mathrm{d}A \wedge \frac{\epsilon_{abmd}}{2}\theta^m \wedge \theta^d\right) + \frac{1}{4}F^{ab}F_{ab}\delta\theta_m \wedge *\theta^m \quad (204)$$

$$= \delta\theta_m \wedge \left(-\frac{1}{4}F^{ab}\mathrm{d}A \wedge \epsilon_{abcd}\eta^{cm}\theta^d + \frac{1}{4}F^{ab}F_{ab}* \theta^m\right) \quad (205)$$

$$= \delta\theta_m \wedge \left(-\frac{1}{4}F^{ab}F_{ef}\epsilon_{abcd}\eta^{cm}\epsilon^{nefd} + \frac{1}{4}\eta^{mn}F^{ab}F_{ab}\right)*\theta_n \quad (206)$$

$$= \delta\theta_m \wedge \left(-\frac{1}{4}F^{ab}F_{ef}\epsilon_{abcd}\eta^{cm}\epsilon^{efnd} + \frac{1}{4}\eta^{mn}F^{ab}F_{ab}\right)*\theta_n \quad (207)$$

$$= \delta\theta_m \wedge \left(F^{ma}F_a^n - \frac{1}{4}\eta^{mn}F^{ab}F_{ab}\right)*\theta_n \quad (208)$$

The second term on the right of equation (199) is,

$$-e\bar{\Psi}\gamma^a A \wedge \delta*\theta_a = -e\bar{\Psi}\gamma^a \Psi A \wedge \delta\left(\frac{\epsilon_{abcd}}{6}\theta^b \wedge \theta^c \wedge \theta^d\right) \quad (209)$$

$$\delta(\epsilon_{abcd}\theta^b \wedge \theta^c \wedge \theta^d) = \epsilon_{abcd}(\delta\theta^b \wedge \theta^c \wedge \theta^d + \theta^b \wedge \delta\theta^c \wedge \theta^d + \theta^b \wedge \theta^c \wedge \delta\theta^d) \quad (210)$$

$$\epsilon_{abcd}\theta^b \wedge \delta\theta^c \wedge \theta^d = -\epsilon_{abcd}\delta\theta^c \wedge \theta^b \wedge \theta^d \quad (211)$$

$$= \epsilon_{acbd}\delta\theta^c \wedge \theta^b \wedge \theta^d \quad (212)$$

$$= \epsilon_{abcd}\delta\theta^b \wedge \theta^c \wedge \theta^d \quad (213)$$

$$-e\bar{\Psi}\gamma^a \Psi A \wedge \delta*\theta_a = -\frac{1}{2}e\bar{\Psi}\gamma^a \Psi A \wedge \epsilon_{abcd}\delta\theta^b \wedge \theta^c \wedge \theta^d \quad (214)$$

$$= \delta\theta^b \wedge \frac{1}{2}e\bar{\Psi}\gamma^a \Psi A \wedge \epsilon_{abcd}\theta^c \wedge \theta^d \quad (215)$$

$$= \delta\theta^m \wedge \frac{1}{2}e\bar{\Psi}\gamma^a \Psi A(v_e)\epsilon_{amcd}\epsilon^{necd}* \theta_n \quad (216)$$

$$= \delta\theta_m \wedge -\frac{1}{2}e\bar{\Psi}\gamma^a\Psi A(v_e)\epsilon_{agcd}\eta^{gm}\epsilon^{ncd} * \theta_n \quad (217)$$

$$= \delta\theta_m \wedge -e\left(\eta^{mn}\bar{\Psi}\gamma^a\Psi A(v_a) - \bar{\Psi}\gamma^n\Psi A(v^m)\right) * \theta_n \quad (218)$$

Similarly, the third term on the right of equation (199) is,

$$i\bar{\Psi}\gamma^a d\Psi \wedge \delta * \theta_a = \delta\theta_m \wedge i\left(\eta^{mn}\bar{\Psi}\gamma^a d\Psi(v_a) - \bar{\Psi}\gamma^n d\Psi(v^m)\right) * \theta_n \quad (219)$$

Using that,

$$\gamma^a\gamma^b\gamma^c d\theta_b(v_a, v_c) = \gamma^b\gamma^a\gamma^c d\theta_a(v_b, v_c) \quad (220)$$

$$-\gamma^a\gamma^b\gamma^c d\theta_c(v_a, v_b) = -\gamma^c\gamma^b\gamma^a d\theta_a(v_c, v_b) \quad (221)$$

$$= -\gamma^b\gamma^c\gamma^a d\theta_a(v_b, v_c) \quad (222)$$

$$= (-2\eta^{ac}\gamma^b + \gamma^b\gamma^a\gamma^c) d\theta_a(v_b, v_c) \quad (223)$$

the fourth term on the right of equation (199) is,

$$\begin{aligned} -\frac{i}{4}\bar{\Psi}\gamma^a\gamma^b\gamma^c\left(\delta(\omega_{bc}(v_a)*1)\right)\Psi &= \\ &= -\frac{i}{8}\bar{\Psi}(\gamma^a\gamma^b\gamma^c + \gamma^b\gamma^a\gamma^c - 2\eta^{ac}\gamma^b + \gamma^b\gamma^a\gamma^c) \times \\ &\quad \times \left(\delta(d\theta_a(v_b, v_c)*1)\right)\Psi \end{aligned} \quad (224)$$

$$= -\frac{i}{8}\bar{\Psi}(2\eta^{ab}\gamma^c - 2\eta^{ac}\gamma^b + \gamma^b\gamma^a\gamma^c)\left(\delta(d\theta_a(v_b, v_c)*1)\right)\Psi \quad (225)$$

$$= -\frac{i}{8}\bar{\Psi}(4\eta^{ab}\gamma^c - \gamma^c\gamma^a\gamma^b)\left(\delta(d\theta_a(v_b, v_c)*1)\right)\Psi \quad (226)$$

The first term on the right of the previous equation is,

$$-\frac{i}{8}\bar{\Psi}4\eta^{ab}\gamma^c\left(\delta(d\theta_a(v_b, v_c)*1)\right)\Psi = \frac{i}{2}\bar{\Psi}\gamma^c\left(\delta(d\theta^a(v_c, v_a)*1)\right)\Psi \quad (227)$$

$$= \frac{i}{2}\bar{\Psi}\gamma^c(\delta d*\theta_c)\Psi \quad (228)$$

$$\begin{aligned} d\left(\frac{i}{2}\bar{\Psi}\gamma^c(\delta*\theta_c)\Psi\right) &= \frac{i}{2}d\bar{\Psi} \wedge \gamma^c(\delta*\theta_c)\Psi + \frac{i}{2}\bar{\Psi}\gamma^c(d\delta*\theta_c)\Psi + \\ &\quad + \frac{i}{2}\bar{\Psi}\gamma^c d\Psi \wedge \delta*\theta_c \end{aligned} \quad (229)$$

so,

$$\frac{i}{2}\bar{\Psi}\gamma^c(d\delta*\theta_c)\Psi = -\frac{i}{2}d\bar{\Psi} \wedge \gamma^c(\delta*\theta_c)\Psi - \frac{i}{2}\bar{\Psi}\gamma^c d\Psi \wedge \delta*\theta_c + d(\dots) \quad (230)$$

Combining equation (219) with the right side of equation (230) gives,

$$\begin{aligned}
& \frac{i}{2} \bar{\Psi} \gamma^a d\Psi \wedge \delta * \theta_a - \frac{i}{2} d\bar{\Psi} \wedge \gamma^a (\delta * \theta_a) \Psi = \\
& = \delta \theta_m \wedge \left(\frac{i}{2} \eta^{mn} \bar{\Psi} \gamma^a d\Psi(v_a) - \frac{i}{2} \bar{\Psi} \gamma^n d\Psi(v^m) - \right. \\
& \quad \left. - \frac{i}{2} \eta^{mn} d\bar{\Psi}(v_a) \gamma^a \Psi + \frac{i}{2} d\bar{\Psi}(v^m) \gamma^n \Psi \right) * \theta_n \quad (231)
\end{aligned}$$

I have,

$$\begin{aligned}
& \left(\frac{i}{2} \eta^{mn} \bar{\Psi} \gamma^a d\Psi(v_a) - \frac{i}{2} \bar{\Psi} \gamma^n d\Psi(v^m) \right)^\dagger = \\
& = -\frac{i}{2} \eta^{mn} d\Psi^\dagger(v_a) \gamma^{a\dagger} \gamma^{0\dagger} \Psi + \frac{i}{2} d\Psi^\dagger(v^m) \gamma^{n\dagger} \gamma^{0\dagger} \Psi \quad (232)
\end{aligned}$$

$$= -\frac{i}{2} \eta^{mn} d\Psi^\dagger(v_a) \gamma^{a\dagger} (-\gamma^0) \Psi + \frac{i}{2} d\Psi^\dagger(v^m) \gamma^{n\dagger} (-\gamma^0) \Psi \quad (233)$$

$$= -\frac{i}{2} \eta^{mn} d\bar{\Psi}(v_a) \gamma^a \Psi + \frac{i}{2} d\bar{\Psi}(v^m) \gamma^n \Psi \quad (234)$$

so the right side of equation (231) becomes,

$$\delta \theta_m \wedge \text{Re} \left(i \eta^{mn} \bar{\Psi} \gamma^a d\Psi(v_a) - i \bar{\Psi} \gamma^n d\Psi(v^m) \right) * \theta_n \quad (235)$$

The second term on the right of equation (226) is,

$$\frac{i}{8} \bar{\Psi} \gamma^c \gamma^a \gamma^b (\delta(d\theta_a(v_b, v_c) * 1)) \Psi = \frac{i}{8} \bar{\Psi} \gamma^c \gamma^a \gamma^b \Psi \delta(d\theta_a \wedge *(\theta_b \wedge \theta_c)) \quad (236)$$

$$= \frac{i}{8} \bar{\Psi} \gamma^c \gamma^a \gamma^b \Psi \left(d\delta\theta_a \wedge *(\theta_b \wedge \theta_c) + d\theta_a \wedge \delta *(\theta_b \wedge \theta_c) \right) \quad (237)$$

I have,

$$\begin{aligned}
& d \left(\bar{\Psi} \gamma^c \gamma^a \gamma^b \Psi (\delta\theta_a \wedge *(\theta_b \wedge \theta_c)) \right) = (d\bar{\Psi}) \gamma^c \gamma^a \gamma^b \Psi \wedge \delta\theta_a \wedge *(\theta_b \wedge \theta_c) + \\
& + \bar{\Psi} \gamma^c \gamma^a \gamma^b d\Psi \wedge \delta\theta_a \wedge *(\theta_b \wedge \theta_c) + \bar{\Psi} \gamma^c \gamma^a \gamma^b \Psi d\delta\theta_a \wedge *(\theta_b \wedge \theta_c) - \\
& - \bar{\Psi} \gamma^c \gamma^a \gamma^b \Psi \delta\theta_a \wedge d *(\theta_b \wedge \theta_c) \quad (238)
\end{aligned}$$

so,

$$\begin{aligned}
& \bar{\Psi} \gamma^c \gamma^a \gamma^b \Psi d\delta\theta_a \wedge *(\theta_b \wedge \theta_c) = \delta\theta_a \wedge \left((d\bar{\Psi}) \gamma^c \gamma^a \gamma^b \Psi \wedge *(\theta_b \wedge \theta_c) + \right. \\
& + \bar{\Psi} \gamma^c \gamma^a \gamma^b d\Psi \wedge *(\theta_b \wedge \theta_c) + \bar{\Psi} \gamma^c \gamma^a \gamma^b \Psi d *(\theta_b \wedge \theta_c) \left. \right) + d(\dots) \quad (239)
\end{aligned}$$

The first term on the right of equation (237) becomes,

$$\begin{aligned} \delta\theta_a \wedge \frac{i}{8} \Big((d\bar{\Psi})\gamma^c\gamma^a\gamma^b\Psi \wedge *(\theta_b \wedge \theta_c) + \bar{\Psi}\gamma^c\gamma^a\gamma^b d\Psi \wedge *(\theta_b \wedge \theta_c) + \\ + \bar{\Psi}\gamma^c\gamma^a\gamma^b\Psi d*(\theta_b \wedge \theta_c) \Big) + d(\dots) \end{aligned} \quad (240)$$

The first term of the previous expression is,

$$\delta\theta_a \wedge \frac{i}{8} d\bar{\Psi}\gamma^c\gamma^a\gamma^b\Psi \wedge *(\theta_b \wedge \theta_c) = \delta\theta_a \wedge \frac{i}{8} (d\bar{\Psi})\gamma^c\gamma^a\gamma^b\Psi \wedge \frac{\epsilon_{bcde}}{2} \theta^d \wedge \theta^e \quad (241)$$

$$= \delta\theta_a \wedge \frac{i}{16} d\bar{\Psi}(v_f)\gamma^c\gamma^a\gamma^b\Psi \epsilon_{bcde} \epsilon^{nfde} * \theta_n \quad (242)$$

$$= \delta\theta_m \wedge \frac{i}{8} d\bar{\Psi}(v_f)(\gamma^f\gamma^m\gamma^n - \gamma^n\gamma^m\gamma^f)\Psi * \theta_n \quad (243)$$

Similarly, formula (240) becomes,

$$\begin{aligned} \delta\theta_m \wedge \left(\frac{i}{8} d\bar{\Psi}(v_f)(\gamma^f\gamma^m\gamma^n - \gamma^n\gamma^m\gamma^f)\Psi * \theta_n + \right. \\ \left. + \frac{i}{8} \bar{\Psi}(\gamma^f\gamma^m\gamma^n - \gamma^n\gamma^m\gamma^f) d\Psi(v_f) * \theta_n + \frac{i}{8} \bar{\Psi}\gamma^c\gamma^m\gamma^b\Psi d*(\theta_b \wedge \theta_c) \right) + d(\dots) \end{aligned} \quad (244)$$

I have,

$$\begin{aligned} \left(\frac{i}{8} d\bar{\Psi}(v_f)(\gamma^f\gamma^m\gamma^n - \gamma^n\gamma^m\gamma^f)\Psi \right)^\dagger = \\ = \frac{-i}{8} \Psi^\dagger (\gamma^{n\dagger}\gamma^{m\dagger}\gamma^{f\dagger} - \gamma^{f\dagger}\gamma^{m\dagger}\gamma^{n\dagger}) \gamma^{0\dagger} d\Psi(v_f) \end{aligned} \quad (245)$$

$$= \frac{-i}{8} \Psi^\dagger (\gamma^{n\dagger}\gamma^{m\dagger}\gamma^{f\dagger} - \gamma^{f\dagger}\gamma^{m\dagger}\gamma^{n\dagger}) (-\gamma^0) d\Psi(v_f) \quad (246)$$

$$= \frac{-i}{8} \Psi^\dagger \gamma^0 (\gamma^n\gamma^m\gamma^f - \gamma^f\gamma^m\gamma^n) d\Psi(v_f) \quad (247)$$

$$= \frac{i}{8} \bar{\Psi} (\gamma^f\gamma^m\gamma^n - \gamma^n\gamma^m\gamma^f) d\Psi(v_f) \quad (248)$$

so formula (244) becomes,

$$\begin{aligned} \delta\theta_m \wedge \left(\text{Re} \left(\frac{i}{4} d\bar{\Psi}(v_f)(\gamma^f\gamma^m\gamma^n - \gamma^n\gamma^m\gamma^f)\Psi \right) * \theta_n + \right. \\ \left. + \frac{i}{8} \bar{\Psi}\gamma^c\gamma^m\gamma^b\Psi d*(\theta_b \wedge \theta_c) \right) + d(\dots) \end{aligned} \quad (249)$$

The second term on the right of equation (237) is,

$$\frac{i}{8}\bar{\Psi}\gamma^c\gamma^a\gamma^b\Psi d\theta_a \wedge \delta*(\theta_b \wedge \theta_c) = \frac{i}{8}\bar{\Psi}\gamma^c\gamma^a\gamma^b\Psi d\theta_a \wedge \delta\left(\frac{\epsilon_{bcde}}{2}\theta^d \wedge \theta^e\right) \quad (250)$$

$$= \frac{i}{8}\bar{\Psi}\gamma^c\gamma^a\gamma^b\Psi d\theta_a \wedge \epsilon_{bcde}\delta\theta^d \wedge \theta^e \quad (251)$$

$$= \frac{i}{8}\bar{\Psi}\gamma^c\gamma^a\gamma^b\Psi d\theta_a \wedge \epsilon_{bcde}\eta^{dm}\delta\theta_m \wedge \theta^e \quad (252)$$

$$= \delta\theta_m \wedge \frac{i}{8}\bar{\Psi}\gamma^c\gamma^a\gamma^b\Psi d\theta_a \wedge \epsilon_{bcde}\eta^{dm}\theta^e \quad (253)$$

The last term of equation (199) is,

$$i\bar{\Psi}m\Psi\delta*1 = im\bar{\Psi}\Psi\delta(\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3) \quad (254)$$

$$= im\bar{\Psi}\Psi(\delta\theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 + \theta^0 \wedge \delta\theta^1 \wedge \theta^2 \wedge \theta^3 + \dots) \quad (255)$$

$$= im\bar{\Psi}\Psi\delta\theta^m \wedge *\theta_m \quad (256)$$

$$= \delta\theta_m \wedge im\bar{\Psi}\Psi\eta^{mn}*\theta_n \quad (257)$$

The variation of the Lagrangean density can be expressed as,

$$\delta\mathfrak{L} = \delta\theta_m \wedge T^m + d(\dots) \quad (258)$$

with T^m being the stress-energy tensor. Gathering terms, it is,

$$\begin{aligned} T^m = & \left(F^{ma}F_a^n - \frac{1}{4}\eta^{mn}F^{ab}F_{ab} - e(\eta^{mn}\bar{\Psi}\gamma^a\Psi A(v_a) - \bar{\Psi}\gamma^n\Psi A(v^m)) \right) + \\ & + \text{Re}(i\eta^{mn}\bar{\Psi}\gamma^a d\Psi(v_a) - i\bar{\Psi}\gamma^n d\Psi(v^m)) + \text{Re}\left(\frac{i}{4}\bar{\Psi}(\gamma^f\gamma^m\gamma^n - \gamma^n\gamma^m\gamma^f)d\Psi(v_f)\right) + \\ & + im\eta^{mn}\bar{\Psi}\Psi\Big)*\theta_n + \frac{i}{8}\bar{\Psi}\gamma^c\gamma^m\gamma^b\Psi d*(\theta_b \wedge \theta_c) + \frac{i}{8}\bar{\Psi}\gamma^c\gamma^a\gamma^b\Psi d\theta_a \wedge \epsilon_{bcde}\eta^{dm}\theta^e \end{aligned} \quad (259)$$

T can also be written as,

$$T^{mn} = -*(\theta^n \wedge T^m) \quad (260)$$

In flat spacetime with the Minkowski metric and with the coframe being the coordinate differentials, T becomes,

$$\begin{aligned} T^{mn} = & F^{ma}F_a^n - \frac{1}{4}\eta^{mn}F^{ab}F_{ab} + \eta^{mn}j^a A_a - j^n A^m + \\ & + \text{Re}(i\eta^{mn}\bar{\Psi}\gamma^a\partial_a\Psi - i\bar{\Psi}\gamma^n\partial^m\Psi) + \\ & + \text{Re}\left(\frac{i}{4}\bar{\Psi}(\gamma^f\gamma^m\gamma^n - \gamma^n\gamma^m\gamma^f)\partial_f\Psi\right) + im\eta^{mn}\bar{\Psi}\Psi \end{aligned} \quad (261)$$

$i\bar{\Psi}\Psi$ is real because γ^0 is a diagonal matrix. Using that,

$$\bar{\Psi}\gamma^a\partial_a\Psi - ij^aA_a = -m\bar{\Psi}\Psi \quad (262)$$

and that,

$$\bar{\Psi}(\gamma^f\gamma^m\gamma^n - \gamma^n\gamma^m\gamma^f)\partial_f\Psi = \bar{\Psi}(2\eta^{fm}\gamma^n - 2\eta^{fn}\gamma^m + [\gamma^m, \gamma^n]\gamma^f)\partial_f\Psi \quad (263)$$

$$\text{Re}\left(\frac{i}{4}\bar{\Psi}[\gamma^m, \gamma^n]\gamma^f\partial_f\Psi\right) = \text{Re}\left(\frac{i}{4}\bar{\Psi}[\gamma^m, \gamma^n](-ie\gamma^aA_a - m)\Psi\right) \quad (264)$$

$$\begin{aligned} &= \text{Re}\left(\frac{1}{4}\bar{\Psi}\left(2e\gamma^mA^n - 2e\gamma^nA^m + \frac{e}{3}[\gamma^m, \gamma^n, \gamma^a]A_a - \right.\right. \\ &\quad \left.\left. - im[\gamma^m, \gamma^n]\right)\Psi\right) \end{aligned} \quad (265)$$

$$\begin{aligned} &= \frac{1}{2}(-j^mA^n + j^nA^m) + \\ &\quad + \text{Re}\left(\bar{\Psi}\left(\frac{e}{12}[\gamma^m, \gamma^n, \gamma^a]A_a - \frac{i}{4}m[\gamma^m, \gamma^n]\right)\Psi\right) \end{aligned} \quad (266)$$

T becomes,

$$\begin{aligned} T^{mn} &= F^{ma}F_a^n - \frac{1}{4}\eta^{mn}F^{ab}F_{ab} - \frac{1}{2}(j^mA^n + j^nA^m) + \\ &\quad + \text{Re}\left(\frac{-i}{2}\bar{\Psi}(\gamma^m\partial^n + \gamma^n\partial^m)\Psi\right) + \\ &\quad + \text{Re}\left(\bar{\Psi}\left(\frac{e}{12}[\gamma^m, \gamma^n, \gamma^a]A_a - \frac{i}{4}m[\gamma^m, \gamma^n]\right)\Psi\right) \end{aligned} \quad (267)$$

This asymmetric stress-energy tensor replicates the known result that a version of the teleparallel equivalent of general relativity is inconsistent in the presence of a Dirac field, in that the left side of the field equation is symmetric and the right side is not. This paragraph applies to the version of TEGR in which a limit is not taken on the model parameters, and both sides of the field equation are derived from a Lagrangean.

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